# **V. Conclusion**

Consider here two very important question for the practical applications of the mathematical physics equations:

* How the mathematical physics problems can be solved, in reality?
* Why the mathematical physics problems are applicable for the practical situation?

## **14. Finite difference method**

We considered before easy enough partial differential equations. The mathematical models of the physical phenomena are described, as a rule, by the difficult enough equations. The classical mathematical physics can be non-applicable for the cases. Therefore, the question arises, how we can solve these problems in the practical situation. Unfortunately, we cannot any possibility to solve these problems analytically, i.e. to find the formula of its analytic solution. However, we can try to find its approximate solution.

We consider only the finite difference method that is universal and easy enough for the realization by the computer. The basis of this method is two ideas. At first, we find the approximate solution of the problem only. Secondary, we do not try to find the solution everywhere. We find it at the concrete finite set of points only.

We start with consideration the one-dimensional analogue of the finite difference method that is the Euler method for the ordinary differential equations. Then, using the formulas of the approximate differentiation, we consider the finite difference method for the boundary problem for the partial differential equations.

### **14.1. Euler method for ordinary differential equations**

Consider the ordinary differential equation

 (14.1)

with initial condition

 (14.2)

We will solve the Cauchy problem (14.1), (14.2) on the interval [0,*T*] by ***Euler method***.

Divide the considered interval by *M* equal parts. Determine the ***step*** *h = T*/*M.* We will consider the solution of the problem (14.1), (14.2) only at the points *ti = ih*, where *i =* 0,1,…,*M.* Consider the equality (14.1) for the value *t = ti*. We have

 (14.3)

Using the definition of the derivative, we find

.

If the value *τ* is small enough, then we obtain the approximate equality



If the number *M* is large enough. Then the step *h* is small enough*.* Put the approximate value of the derivative to the equality (14.3). We obtain



Note that *ti + h = ti*+1. Denote *xi* = *x*(*ti*). We get

 (14.4)

The value *x*0 here is known from the initial condition (14.2). Then we can find all value *xi* by the formula (14.4). Thus, we find the approximate solution of the problem (14.1), (14.2) at the considered points. This solution is not exact, because we approximate the derivative of the given equation by the corresponding difference.

The Euler method is applicable for the Cauchy problem for the system of the first order ordinary differential equations. In this situation the system (14.1), (14.2) has the sense in the vector form. Its approximate solution can be found by the formula (14.4) with vector interpretation. We know that the high order differential equation can be transformed to the system of the first order ordinary differential equations. Therefore, the Euler method is applicable too for the Cauchy problem for high order ordinary differential equations.

### **14.2. Approximation of derivatives**

The basis of the considered method is the approximation of the derivative. Consider the method of approximation of function derivative. The corresponding theory uses ***Taylor formula***. Consider the smooth enough function *f = f*(*y*). We have the Taylor formula

 (14.5)

If the number *a* is small enough, we can break the sum at the right-hand side of this equality.

If we use two summands, we obtain



where *o*(*a*)/*a* → 0 as *a* → 0. Now we find

 (14.6)

This approximation formula is called the ***forward difference***.

Sometimes it will be preferable to use the negative value of the step. Then we have the formula



Then we get

**** (14.7)

This is the ***back difference*** formula.

For both formulas do not use the term *o*(*a*)/*a* that has the first order of smallness with respect to the step *a*. One says that the formulas have ***first order of approximation***. We can use also more exact formulas, if we consider more summands of the Taylor formula. If we use three summands, we get



where *o*(*a*2)/*a*2→ 0 as *a* → 0. Change the sigh before *a* at this formula. We obtain



After difference second equality from the first one, we determine



Note that the value  for the formula are different. We have an interest to its order of smallness only. Then after its difference, we obtain the term with same order. Now we find

 (14.8)

This formula is called the ***central difference***. It has the ***second order of approximation***, because the value *o*(*a*2)/*a* has second order of smallness with respect to *a.*

We consider the second order partial differential equations. Therefore, it is necessary to have the approximation formulas for the second derivative too. Use four summands at the Taylor formula. We have



where *o*(*a*3)/*a*3→ 0 as *a* → 0. Write the analogical formula with negative step. Determine



Add these equalities. We get



Now we find

 (14.9)

This is the general formula for the approximation of the second derivative. It has too the second order of approximation, because the value *o*(*a*3)/*a*2has second order of smallness with respect to the step *a.*

We use the obtained formula for solving the boundary problems for the mathematical physics equations.

### **14.3. Finite difference method for the heat equation (first boundary problem)**

Consider the first boundary problem for the heat equation. We have the equation

*ut* = *a*2 *uxx + f*(*x*,*t*), 0<*x<L*, 0<*t*<*T* (14.10)

with initial condition

*u*(*x*,0) = *ϕ*(*x*) , 0<*x<L* (14.11)

and boundary conditions

*u*(0,*t*) = *p*(*t*), *u*(*L*,*t*) = *q*(*t*), 0<*t*<*T*, (14.12)

where *f*, *ϕ*, *p* and *q* are known functions.

Divide the interval [0,*L*] by *M* equal parts, and the interval [0,*T*] by *N* equal parts. Denote   
*h = L*/*M*, *τ = T*/*N.* Determine the points

*xi =ih*, *i =* 0,1,…,*M*; *yj = jτ*, *j =* 0,1,…,*N.*

Consider the equation (14.10) at the arbitrary point (*xi*,*tj*). We have

*ut*(*xi*,*tj*) = *a*2 *uxx*(*xi*,*tj*) *+ f*(*xi*,*tj*), (14.13)

Use the forward difference formula (14.6) for the approximate the first derivative with respect to *t* and the formula (14.9) with respect to *x.* We get



Denote  Then we have



Find the value

 (14.14)

By this formula, we can determine the value , if we know  How we can determine it?

We have also the initial and boundary conditions. From the equality (14.11), we can find

 (14.15)

where  Then from the equalities (14.12), determine

 (14.16)

where 

Now we have the following algorithm

1. Determine the values  from the equality (14.15) for *i =* 0,…,*M*.
2. Determine the values  and  from the equality (14.15) for *j =* 0.
3. Calculate the values  by the formula (14.14) for *j* = 0, *i =* 1,…,*M*–1, using the known values ,  and .
4. Calculate the values  by the formula (14.14) for *j* = 1, *i =* 1,…,*M*–1, using the known values ,  and .
5. Repeat the calculation for all numbers *j<N*.

### **14.4. Finite difference method for the heat equation (second boundary problem)**

Consider the second boundary problem for the heat equation. We have the equation

*ut* = *a*2*uxx + f*(*x*,*t*), 0<*x<L*, 0<*t*<*T* (14.17)

with initial condition

*u*(*x*,0) = *ϕ*(*x*) , 0<*x<L* (14.18)

and boundary conditions

*ux*(0,*t*) = *p*(*t*), *ux*(*L*,*t*) = *q*(*t*), 0<*t*<*T*, (14.19)

where *f*, *ϕ*, *p* and *q* are known functions.

Divide again the interval [0,*L*] by *M* equal parts, and the interval [0,*T*] by *N* equal parts with using the previous denotations. We have again the formula

 (14.20)

that is the equality (14.14). From the initial condition, it follows the equality

 (14.21)

that is the equality (14.15).

The unique difference between the boundary problems (14.10) – (14.12) and (14.17) – (14.19) are the boundary conditions. Approximate the first boundary condition (14.19), using the forward difference formula (14.6). We get



Find the value

 (14.22)

Use the back difference formula (14.7) for the approximation the second boundary condition (14.19). We have



Determine

 (14.23)

Now we have the following algorithm

1. Determine the values  from the equality (14.21) for *i =* 0,…,*M*.
2. Calculate the values  by the formula (14.20) for *j* = 0, *i =* 1,…,*M*–1, using the known values  for all *i*.
3. Find the values  by the formula (14.22), using the known value 
4. Find the values  by the formula (14.23), using the known value 
5. Calculate the values  by the formula (14.20) for *j* = 1, *i =* 1,…,*M*–1, using the known values  for all *i*.
6. Find the values  by the formula (14.22), using the known value 
7. Find the values  by the formula (14.23), using the known value 
8. Repeat the calculation for all value *j<N*.

### **14.5. Finite difference method for the vibrating string equation**

Consider the first boundary problem for the vibrating string equation. We have the equation

*utt* = *a*2 *uxx + f*(*x*,*t*), 0<*x<L*, 0<*t*<*T*

with initial conditions

*u*(*x*,0) = *ϕ*(*x*), *ut*(*x*,0) = *ψ*(*x*), 0<*x<L*

and boundary conditions

*u*(0,*t*) = *p*(*t*), *u*(*L*,*t*) = *q*(*t*), 0<*t*<*T*,

where *f*, *ϕ*, *ψ*, *p* and *q* are known functions.

Divide the interval [0,*L*] by *M* equal parts, and the interval [0,*T*] by *N* equal parts. Denote   
*h = L*/*M*, *τ = T*/*N.* Determine the points

*xi =ih*, *i =* 0,1,…,*M*; *yj = jτ*, *j =* 0,1,…,*N.*

Consider the equation at the arbitrary point (*xi*,*tj*). We have

*utt*(*xi*,*tj*) = *a*2 *uxx*(*xi*,*tj*) *+ f*(*xi*,*tj*),

Use the formula (14.9) with respect to *t* and *x.*



Denote  Then we have



Find the value



By this formula, we can determine the value , if we know  How we can determine?

From first initial condition, we get



Approximating the second initial condition, we obtain



Find



Using the boundary conditions, determine



Now we have the following algorithm

1. Determine the values  for *i =* 0,…,*M*.
2. Determine the values  for *i =* 0,…,*M*.
3. Determine the values  and  for *j =* 1.
4. Calculate the values  *i =* 1,…,*M*–1
5. Repeat the calculation for all numbers *j<N*.

### **Conclusions**

* The difficult enough mathematical physics problems can be solved approximately with using the computer.
* The easiest numerical method of solving the Cauchy problem for ordinary differential equation is the Euler method.
* The Euler method is based on the approximation of derivative.
* There exists many formulas of derivative approximations with different orders of approximations.
* The finite different method is an extension of the Euler method to the partial differential equations.
* The finite different method uses the approximation of all derivatives including to the equation and boundary conditions.
* We apply this method for finding an approximate solutions of the boundary problem for heat equation and vibrating string equation

### **Task. Finite difference method**

Consider the extended heat equation with initial condition

*u*(*x*,0) = *ϕ*(*x*), 0<*x<L*

and different variant of equation and the boundary conditions.

Table of parameter

|  |  |  |  |
| --- | --- | --- | --- |
| **variant** | **equation** | **left boundary condition** | **right boundary condition** |
| **1** | *ut* = *a*2*uxx +ux* + *f*(*x*,*t*) | *u*(0,*t*) = *p*(*t*) | *ux*(*L*,*t*) = -*q*(*t*) |
| **2** | *ut* = *a*2*uxx +u* + *f*(*x*,*t*) | *u*(0,*t*) = -*p*(*t*) | *ux*(*L*,*t*) = *q*(*t*) |
| **3** | *ut* = *a*2*uxx - ux* + *f*(*x*,*t*) | *u*(0,*t*) = *p*(*t*) | *u*(*L*,*t*) = -*q*(*t*) |
| **4** | *ut* = *a*2*uxx - u* + *f*(*x*,*t*) | *u*(0,*t*) = -*p*(*t*) | *u*(*L*,*t*) = *q*(*t*) |
| **5** | *ut* = *a*2*uxx - ux* + *f*(*x*,*t*) | *ux*(0,*t*) = *p*(*t*) | *u*(*L*,*t*) = -*q*(*t*) |
| **6** | *ut* = *a*2*uxx +u* + *f*(*x*,*t*) | *ux*(0,*t*) = -*p*(*t*) | *u*(*L*,*t*) = *q*(*t*) |
| **7** | *ut* = *a*2*uxx +ux* + *f*(*x*,*t*) | *ux*(0,*t*) = *p*(*t*) | *ux*(*L*,*t*) = -*q*(*t*) |
| **8** | *ut* = *a*2*uxx - u* + *f*(*x*,*t*) | *ux*(0,*t*) = -*p*(*t*) | *ux*(*L*,*t*) = *q*(*t*) |

Steps of the answer

1. Give the completely problem statement.
2. Give the approximation of the equation, initial condition and boundary conditions.
3. Write the completely algorithm of the finite difference method.